Lösungen zur Algebra, S. 57f

1. a) $a^2 = \frac{A}{6} \quad \Leftrightarrow \quad a = \pm\sqrt{\frac{A}{6}}$ b) $a^2 = \frac{3V}{h} \quad \Leftrightarrow \quad a = \pm\sqrt{\frac{3V}{h}}$

c) $v^2 = \frac{2E}{m} \quad \Leftrightarrow \quad v = \pm\sqrt{\frac{2E}{m}}$ d) $T^2 = 4\pi^2 \frac{\ell}{g} \quad \Leftrightarrow \quad T = \pm 2\pi\sqrt{\frac{\ell}{g}}$

e) $h^2 = a^2 - \frac{1}{4}a^2 = \frac{3}{4}a^2 \quad \Leftrightarrow \quad h = \pm\frac{1}{2}\sqrt{3}\cdot a$

f) $9r^2 = a^2 - \frac{1}{4}a^2 = \frac{3}{4}a^2 \quad \Leftrightarrow \quad r = \pm\frac{1}{2\sqrt{3}}a = \pm\frac{1}{6}\sqrt{3}\cdot a$

g) $\frac{9}{4}r^2 = a^2 - \frac{1}{4}a^2 = \frac{3}{4}a^2 \quad \Leftrightarrow \quad r = \pm\frac{1}{\sqrt{3}}a = \pm\frac{1}{3}\sqrt{3}\cdot a$

h) $\frac{4}{9}h^2 - \frac{1}{9}h^2 = \frac{1}{4}a^2 \quad \Leftrightarrow \quad \frac{1}{3}h^2 = \frac{1}{4}a^2 \quad \Leftrightarrow \quad a = \pm\sqrt{\frac{4}{3}h^2} = \pm\frac{2}{3}\sqrt{3}\cdot h$

2. a) $x^2 - bx + ax - ab = 0 \quad \Leftrightarrow \quad x^2 + (a-b)x - ab = 0 \quad \Rightarrow$

$x_{1;2} = \frac{-(a-b)\pm\sqrt{(b-a)^2+4ab}}{2} = \frac{-a+b\pm\sqrt{b^2+2ab+a^2}}{2} = \frac{-a+b\pm|b+a|}{2}$

$x_1 = \frac{-a+b+a+b}{2} = b \qquad x_2 = \frac{-a+b-a-b}{2} = -a \qquad \mathbb{L} = \{-a;\ b\}$

b) $6x^2 - 3xb = ab - 2ax \quad \Leftrightarrow \quad 6x^2 + (2a-3b)x - ab = 0 \quad \Rightarrow$

$x_{1;2} = \frac{-(2a-3b)\pm\sqrt{(2a-3b)^2+24ab}}{12} = \frac{-2a+3b\pm\sqrt{4a^2+12ab+9b^2}}{12} = \frac{-2a+3b\pm|2a+3b|}{12}$

$x_1 = \frac{-2a+3b+2a+3b}{12} = \frac{1}{2}b \qquad x_2 = \frac{-2a+3b-2a-3b}{12} = -\frac{1}{3}a \qquad \mathbb{L} = \{-\frac{1}{3}a;\ \frac{1}{2}b\}$

c) $x^2 - (a+b)x + ab = 0 \quad \Rightarrow \quad x_{1;2} = \frac{a+b\pm\sqrt{[-(a+b)]^2-4ab}}{2} = \frac{a+b\pm\sqrt{a^2-2ab+b^2}}{2} = \frac{a+b\pm|a-b|}{2}$

$x_1 = \frac{a+b+a-b}{2} = a \qquad x_2 = \frac{a+b-a+b}{2} = b \qquad \mathbb{L} = \{a;\ b\}$

d) $x^2 + (n-m)x - mn = 0 \quad \Rightarrow \quad x_{1;2} = \frac{-(n-m)\pm\sqrt{(n-m)^2+4mn}}{2} = \frac{m-n\pm\sqrt{m^2+2mnb+n^2}}{2} = \frac{m-n\pm|m+n|}{2}$

$x_1 = \frac{m-n+m+n}{2} = m \qquad x_2 = \frac{m-n-m-n}{2} = -n \qquad \mathbb{L} = \{m;\ -n\}$

e) $(x+3k)^2 = 0 \quad \Rightarrow \quad x + 3k = 0 \quad \Leftrightarrow \quad x = -3k \qquad \mathbb{L} = \{-3k\}$

f) $x^2 - 3ax + 2a^2 = 0 \quad \Rightarrow \quad x_{1;2} = \frac{3a\pm\sqrt{9a^2-8a^2}}{2} = \frac{3a\pm\sqrt{a^2}}{2} = \frac{3a\pm|a|}{2}$

$x_1 = \frac{3a+a}{2} = 2a \qquad x_2 = \frac{3a-a}{2} = a \qquad \mathbb{L} = \{a;\ 2a\}$

3. a) $x^2 - 2ax + a^2 - 1 = 0 \quad \Rightarrow \quad x_{1;2} = \frac{2a\pm\sqrt{4a^2-4(a^2-1)}}{2} = \frac{2a\pm\sqrt{4}}{2} = \frac{2a\pm 2}{2}$

$x_1 = \frac{2a+2}{2} = a + 1 \qquad x_2 = \frac{2a-2}{2} = a - 1 \qquad \mathbb{L} = \{a + 1;\ a - 1\}$

b) $x^2 - 2ax + a^2 - b^2 = 0 \quad \Rightarrow \quad x_{1;2} = \frac{2a\pm\sqrt{4a^2-4(a^2-b^2)}}{2} = \frac{2a\pm\sqrt{4b^2}}{2} = \frac{2a\pm 2\cdot|b|}{2}$

$x_1 = \frac{2a+2b}{2} = a + b \qquad x_2 = \frac{2a-2b}{2} = a - b \qquad \mathbb{L} = \{a + b;\ a - b\}$

c) $x^2 - (3a+1)x + 2a^2 + a = 0 \Rightarrow x_{1;2} = \frac{3a+1\pm\sqrt{[-(3a+1)]^2-4(2a^2+a)}}{2} = \frac{3a+1\pm\sqrt{a^2+2a+1}}{2} = \frac{3a+1\pm 2\cdot|a+1|}{2}$

$x_1 = \frac{3a+1+a+1}{2} = 2a + 1 \qquad x_2 = \frac{3a+1-a-1}{2} = a \qquad \mathbb{L} = \{2a + 1;\ a\}$

d) $x^2 - (4a+1)x + 3(a^2+a) = 0 \Rightarrow x_{1;2} = \frac{4a+1\pm\sqrt{[-(4a+1)]^2-12(a^2+a)}}{2} = \frac{4a+1\pm\sqrt{4a^2-4a+1}}{2} = \frac{4a+1\pm|2a-1|}{2}$

$x_1 = \frac{4a+1+2a-1}{2} = 3a \qquad x_2 = \frac{4a+1-2a+1}{2} = a + 1 \qquad \mathbb{L} = \{3a;\ a + 1\}$

e) $x^2 + 2bx + 2ab - a^2 = 0 \quad \Rightarrow \quad x_{1;2} = \frac{-2b\pm\sqrt{4b^2-8ab+4a^2}}{2} = \frac{-2b\pm 2\sqrt{a^2-2ab+b^2}}{2} = \frac{-2b\pm 2\cdot|a-b|}{2}$

$x_1 = \frac{-2b+2(a-b)}{2} = a - 2b \qquad x_2 = \frac{-2b-2(a-b)}{2} = -a \qquad \mathbb{L} = \{ä - 2b;\ -a\}$

f) $x^2 - 2(1+c)x + c^2 + 2c = 0 \quad \Rightarrow \quad x_{1;2} = \frac{2(1+c)\pm\sqrt{[-2(1+c)]^2-4c^2-8c}}{2} = \frac{2(1+c)\pm\sqrt{4}}{2} = \frac{2+2c\pm 2}{2}$

$x_1 = \frac{2+2c+2}{2} = 2 + c \qquad x_2 = \frac{2+2c-2}{2} = c \qquad \mathbb{L} = \{2 + c;\ c\}$

4. a) $ax^2-(a+1)x+1=0 \quad\Rightarrow\quad x_{1;2}=\frac{a+1\pm\sqrt{[-(a+1)]^2-4a}}{2a}=\frac{a+1\pm\sqrt{a^2-2a+1}}{2a}=\frac{a+1\pm|a-1|}{2a}$

$x_1=\frac{a+1+a-1}{2a}=1 \qquad x_2=\frac{a+1-a+1}{2a}=\frac{1}{a} \qquad \mathbb{L}=\{1;\ \frac{1}{a}\};\ a\neq 0$

b) $a^2x^2+2ax-15=0 \quad\Rightarrow\quad x_{1;2}=\frac{-2a\pm\sqrt{(2a)^2+60a^2}}{2a^2}=\frac{-2a\pm\sqrt{64a^2}}{2a^2}=\frac{-2a\pm 8|a|}{2a^2}$

$x_1=\frac{-2a+8a}{2a^2}=\frac{3}{a} \qquad x_2=\frac{-2a-8a}{2a^2}=-\frac{5}{a} \qquad \mathbb{L}=\{\frac{3}{a};\ -\frac{5}{a}\};\ a\neq 0$

c) $4x^2-2(2c+d)x+c^2+dc-2d^2=0 \quad\Rightarrow$

$x_{1;2}=\frac{2(2c+d)\pm\sqrt{[-2(2c+d)]^2-16c^2-16cd+32d^2}}{8}=\frac{2(2c+d)\pm\sqrt{36d^2}}{8}=\frac{2(2c+d)\pm 6\cdot|d|}{8}$

$x_1=\frac{4c+2d+6d}{8}=\frac{1}{2}c+d \qquad x_2=\frac{4c+2d-6d}{8}=\frac{1}{2}c-\frac{1}{2}d \qquad \mathbb{L}=\{\frac{1}{2}c+d;\ \frac{1}{2}c-\frac{1}{2}d\}$

d) $x^2-(a+b)x-2(a-b)^2+ab=0 \quad\Rightarrow$

$x_{1;2}=\frac{a+b\pm\sqrt{[-(a+b)]^2+8(a-b)^2-4ab}}{2}=\frac{a+b\pm\sqrt{9a^2-18ab+9b^2}}{2}=\frac{a+b\pm 3|a-b|}{2}$

$x_1=\frac{a+b+3a-3b}{2}=2a-b \qquad x_2=\frac{a+b-3a+3b}{2}=-a+2b \qquad \mathbb{L}=\{2a-b;\ -a+2b\}$

e) $t^2x^2-2t^2\cdot x+9+t^2=0 \quad\Rightarrow\quad x_{1;2}=\frac{2t^2\pm\sqrt{(-2t^2)^2-36t^2-4t^4}}{2t^2}=\frac{2t^2\pm\sqrt{-36t^2}}{2t^2} \quad\Rightarrow\quad \mathbb{L}=\{\ \}$

f) $a^2x^2+6ax+9-4a^2=0 \quad\Rightarrow\quad x_{1;2}=\frac{-6a\pm\sqrt{36a^2-36a^2+16a^4}}{2a^2}=\frac{-6a\pm 4a^2}{2a^2}$

$x_1=\frac{-6a+4a^2}{2a^2}=-\frac{3}{a}+2 \qquad x_2=\frac{-6a-4a^2}{2a^2}=-\frac{3}{a}-2 \qquad \mathbb{L}=\{-\frac{3}{a}+2;\ -\frac{3}{a}-2\}$

5. a) $\mathbb{D}=\mathbb{R}\setminus\{0\} \qquad x^2-2a^2=ax \quad\Leftrightarrow\quad x^2-ax-2a^2=0 \quad\Rightarrow\quad x_{1;2}=\frac{a\pm\sqrt{a^2+8a^2}}{2}=\frac{a\pm 3\cdot|a|}{2}$

$x_1=\frac{a+3a}{2}=2a \qquad x_2=\frac{a-3a}{2}=-a \qquad \mathbb{L}=\{2a;\ -a\}$

b) $\mathbb{D}=\mathbb{R}\setminus\{b\} \qquad (x+2)(x-b)-2b=2b(x-b) \quad\Leftrightarrow\quad x^2+(2-3b)x+2b^2-4b=0 \quad\Rightarrow$

$x_{1;2}=\frac{3b-2\pm\sqrt{(2-3b)^2-8b^2+16b}}{2}=\frac{3b-2\pm\sqrt{b^2+4b+4}}{2}=\frac{3b-2\pm|b+2|}{2}$

$x_1=\frac{3b-2+b+2}{2}=2b \qquad x_2=\frac{3b-2-b-2}{2}=b-2 \qquad \mathbb{L}=\{2b;\ b-2\}$

c) $\mathbb{D}=\mathbb{R}\setminus\{-c;\ c\} \quad 7c(x-c)-c(x+c)=\frac{4}{5}(x+c)(x-c) \quad\Leftrightarrow\quad 2x^2-15cx+18c^2=0 \quad\Rightarrow$

$x_{1;2}=\frac{15c\pm\sqrt{225c^2-144c^2}}{2}=\frac{15c\pm\sqrt{81c^2}}{2}=\frac{15c\pm 9\cdot|c|}{2}$

$x_1=\frac{15c+9c}{2}=12c \qquad x_2=\frac{15c-9c}{2}=6c \qquad \mathbb{L}=\{12c;\ 6c\}$

d) $\mathbb{D}=\mathbb{R}\setminus\{a;\ -a\} \quad x(x-a)+6a^2=3a(x+a) \quad\Leftrightarrow\quad x^2-4ax+3a^2=0 \quad\Rightarrow$

$x_{1;2}=\frac{4a\pm\sqrt{16a^2-12a^2}}{2}=\frac{4a\pm 2\cdot|a|}{2} \qquad x_1=\frac{4a+2a}{2}=3a \qquad x_2=\frac{4a-2a}{2}=a \qquad \mathbb{L}=\{3a;\ a\}$

6. a) $D = 36 - 4t$ — 2 Lösungen für $D > 0$, d.h. für $t < 9$
1 Lösung für $D = 0$, d.h. für $t = 9$ — 0 Lösungen für $D < 0$, d.h. für $t > 9$

b) $D = 9t^2 + 72$ — $D > 0$ für alle Werte von t, also gibt es immer 2 Lösungen

c) $D = (2t - 1)^2 - 4t^2 = -4t + 1$ — 2 Lösungen für $D > 0$, d.h. für $t < \frac{1}{4}$
1 Lösung für $D = 0$, d.h. für $t = \frac{1}{4}$ — 0 Lösungen für $D < 0$, d.h. für $t > \frac{1}{4}$

d) $D = 4 - 8t$ — 2 Lösungen für $D > 0$, d.h. für $t < \frac{1}{2}$
1 Lösung für $D = 0$, d.h. für $t = \frac{1}{2}$ — 0 Lösungen für $D < 0$, d.h. für $t > \frac{1}{2}$

e) $D = (4 + 2t)^2 - 4t^2 = 16 + 16t$ — 2 Lösungen für $D > 0$, d.h. für $t > -16$
1 Lösung für $D = 0$, d.h. für $t = -16$ — 0 Lösungen für $D < 0$, d.h. für $t < -16$

f) $D = (3 - 2t)^2 - 4t^2 = 9 - 12t$ — 2 Lösungen für $D > 0$, d.h. für $t < \frac{3}{4}$
1 Lösung für $D = 0$, d.h. für $t = \frac{3}{4}$ — 0 Lösungen für $D < 0$, d.h. für $t > \frac{3}{4}$

7. a) D = 16 – 16 + 4a = 4a 2 Lösungen für D > 0, d.h. für a > 0

1 Lösung für D = 0, d.h. für a = 0 0 Lösungen für D < 0, d.h. für a < 0

$x_{1;2} = \frac{4 \pm \sqrt{4a}}{2} = \frac{4 \pm 2\sqrt{a}}{2} = 2 \pm \sqrt{a}$ $x_1 = 2 + \sqrt{a}$ $x_2 = -3 - \sqrt{-b}$

Vieta-Probe: $x_1 + x_2 = 2 + \sqrt{a} + 2 - \sqrt{a} = 4$ $x_1 \cdot x_2 = \left(2 + \sqrt{a}\right) \cdot \left(2 - \sqrt{a}\right) = 4 - a$

b) D = 36 – 36 – 4b = –4b 2 Lösungen für D > 0, d.h. für b < 0

1 Lösung für D = 0, d.h. für b = 0 0 Lösungen für D < 0, d.h. für b > 0

$x_{1;2} = \frac{-6 \pm \sqrt{-4b}}{2} = \frac{-6 \pm 2\sqrt{-b}}{2} = -3 \pm \sqrt{-b}$ $x_1 = -3 + \sqrt{-b}$ $x_2 = 2 - \sqrt{a}$

Vieta-Probe: $x_1 + x_2 = -3 + \sqrt{-b} + \left(-3 - \sqrt{-b}\right) = -6$

$x_1 \cdot x_2 = \left(-3 + \sqrt{-b}\right) \cdot \left(-3 - \sqrt{-b}\right) = 9 + b$

c) D = 9c – 8c = c 2 Lösungen für D > 0, d.h. für c > 0

1 Lösung für D = 0, d.h. für c = 0 0 Lösungen für D < 0, d.h. für c < 0

$x_{1;2} = \frac{3\sqrt{c} \pm \sqrt{c}}{2}$ $x_1 = \frac{3\sqrt{c} + \sqrt{c}}{2} = 2\sqrt{c}$ $x_2 = \frac{3\sqrt{c} - \sqrt{c}}{2} = \sqrt{c}$

Vieta-Probe: $x_1 + x_2 = 2\sqrt{c} + \sqrt{c} = 3\sqrt{c}$ $x_1 \cdot x_2 = 2\sqrt{c} \cdot \sqrt{c} = 2c$

d) D = 4a – 4a + 4b = 4b 2 Lösungen für D > 0, d.h. für b > 0

1 Lösung für D = 0, d.h. für b = 0 0 Lösungen für D < 0, d.h. für b < 0

$x_{1;2} = \frac{-2\sqrt{a} \pm \sqrt{4b}}{2} = \frac{-2\sqrt{a} \pm 2\sqrt{b}}{2} = -\sqrt{a} \pm \sqrt{b}$ $x_1 = -\sqrt{a} + \sqrt{b}$ $x_2 = -\sqrt{a} - \sqrt{b}$

Vieta-Probe: $x_1 + x_2 = -\sqrt{a} + \sqrt{b} - \sqrt{a} - \sqrt{b} = -2\sqrt{a}$

$x_1 \cdot x_2 = \left(-\sqrt{a} + \sqrt{b}\right) \cdot \left(-\sqrt{a} - \sqrt{b}\right) = a - b$

e) D = 4t – 12t = –8t 2 Lösungen für D > 0, d.h. für t < 0

1 Lösung für D = 0, d.h. für t = 0 0 Lösungen für D < 0, d.h. für t > 0

$x_{1;2} = \frac{2\sqrt{t} \pm \sqrt{-8t}}{2} = \frac{2\sqrt{t} \pm 2\sqrt{-2t}}{2} = \sqrt{t} \pm \sqrt{-2t}$ $x_1 = \sqrt{t} + \sqrt{-2t}$ $x_2 = \sqrt{t} - \sqrt{-2t}$

Vieta-Probe: $x_1 + x_2 = \sqrt{t} + \sqrt{-2t} + \sqrt{t} - \sqrt{-2t} = 2\sqrt{t}$

$x_1 \cdot x_2 = \left(\sqrt{t} + \sqrt{-2t}\right) \cdot \left(\sqrt{t} - \sqrt{-2t}\right) = t - (-2t) = 3t$

f) D = 4k – 4(2k – 4) = –4k + 16 2 Lösungen für D > 0, d.h. für k < 4

1 Lösung für D = 0, d.h. für k = 4 0 Lösungen für D < 0, d.h. für k > 4

$x_{1;2} = \frac{2\sqrt{k} \pm \sqrt{-4k+16}}{2} = \frac{2\sqrt{k} \pm 2\sqrt{-k+4}}{2} = \sqrt{k} \pm \sqrt{-k+4}$

$x_1 = \sqrt{k} + \sqrt{-k+4}$ $x_2 = \sqrt{k} - \sqrt{-k+4}$

Vieta-Probe: $x_1 + x_2 = \sqrt{k} + \sqrt{-k+4} + \sqrt{k} - \sqrt{-k+4} = 2\sqrt{k}$

$x_1 \cdot x_2 = \left(\sqrt{k} + \sqrt{-k+4}\right) \cdot \left(\sqrt{k} - \sqrt{-k+4}\right) = k - (-k+4) = 2k - 4$

8. a) $D = t^2 - 36$ $D = 0 \Rightarrow t^2 - 36 = 0 \Leftrightarrow t^2 = 36 \Leftrightarrow t = \pm 6$

1. Fall: t = 6 $x^2 - 6x + 9 = 0 \Leftrightarrow (x-3)^2 = 0$ $\mathbb{L} = \{3\}$

2. Fall: t = –6 $x^2 + 6x + 9 = 0 \Leftrightarrow (x+3)^2 = 0$ $\mathbb{L} = \{-3\}$

b) $D = (-3-3t)^2 - 8t = 9 + 10t + 9t^2$

$D = 0 \Rightarrow t_{1;2} = \frac{-10 \pm \sqrt{100-324}}{2} = \frac{-10 \pm \sqrt{-224}}{2}$

$\Rightarrow$ es gibt keinen Wert für t, für den die Gleichung nur eine Lösung hat.

c) $D = 100 - 4t^2$ $D = 0 \Rightarrow 100 - 4t^2 = 0 \Leftrightarrow t^2 = 25 \Leftrightarrow t = \pm 5$

1. Fall: t = 5 $5x^2 + 10x + 5 = 0 \Leftrightarrow 5(x+1)^2 = 0$ $\mathbb{L} = \{-1\}$

2. Fall: t = –5 $-x^2 + 10x - 5 = 0 \Leftrightarrow -5(x-1)^2 = 0$ $\mathbb{L} = \{1\}$

d) $D = (2-t)^2 - 4t^2 = 4 - 4t - 3t^2$ $D = 0 \Rightarrow t_{1;2} = \frac{4 \pm \sqrt{16+48}}{-6} = \frac{4 \pm 8}{-6}$

1. Fall: $t_1 = \frac{4+8}{-6} = -2$ $-2x^2 + 2x + 2x - 2 = 0 \Leftrightarrow -2(x-1)^2 = 0$ $\mathbb{L} = \{1\}$

2. Fall: $t_1 = \frac{4-8}{-6} = \frac{2}{3}$ $\frac{2}{3}x^2 - \frac{2}{3}x + 2x + \frac{2}{3} = 0 \Leftrightarrow \frac{2}{3}(x+1)^2 = 0$ $\mathbb{L} = \{-1\}$

9. a) $\frac{a-x}{x} = \frac{x}{a} \Leftrightarrow \frac{a}{x} - 1 = \frac{x}{a} \Leftrightarrow a^2 - ax = x^2 \Leftrightarrow x^2 + ax - a^2 = 0 \Rightarrow x_{1;2} = \frac{-a \pm \sqrt{a^2 + 4a^2}}{2} = \frac{-a \pm \sqrt{5a^2}}{2}$

Es gilt nur die 1. Lösung, da Längen positiv sind: $x = \frac{-a + a\sqrt{5}}{2}$

b) $\frac{x}{a} = \frac{-a + a\sqrt{5}}{2a} = \frac{-1 + \sqrt{5}}{2} = \frac{-4 + \sqrt{80}}{8} \approx \frac{-4+9}{8} = \frac{5}{8}$ c) $\frac{5}{8} \cdot 160\,m = 100\,m$