Lösungen:

1. a) $f_k(x) = \frac{1+k\cdot\ln x}{x}$ $\quad \mathbb{D} = \mathbb{R}^+ \quad k \in \mathbb{R}^+$

Nullstelle für $1+k\cdot\ln x = 0$

$$\ln x = -\frac{1}{k}$$

$$x = e^{-\frac{1}{k}} \qquad \left(e^{-\frac{1}{k}} \middle| 0\right)$$

$$\lim_{x\to+\infty} \frac{1+k\cdot\ln x}{x} = \lim_{x\to+\infty}\left(\frac{1}{x} + k\cdot\frac{\ln x}{x}\right) = 0^+ + k\cdot 0^+ = 0^+ \quad \text{(siehe Hinweis)}$$

$$\lim_{x \overset{>}{\to} 0} \frac{1+k\cdot\ln x}{x} = \lim_{x \overset{>}{\to} 0} \frac{1}{x}(1+k\cdot\ln x) = (+\infty)\cdot(-\infty) = -\infty$$

b) $f_k'(x) = \frac{k\cdot\frac{1}{x}\cdot x - (1+k\cdot\ln x)\cdot 1}{x^2} = \frac{k-1-k\cdot\ln x}{x^2}$

$$f_k''(x) = \frac{-k\cdot\frac{1}{x}\cdot x^2 - (k-1-k\cdot\ln x)\cdot 2x}{x^4} = \frac{-k-2k+2+2k\cdot\ln x}{x^3} = \frac{2-3k+2k\cdot\ln x}{x^3}$$

$f_k'(x) = 0$

$k-1-k\cdot\ln x = 0$

$$\ln x = \frac{k-1}{k}$$

$$x = e^{\frac{k-1}{k}}$$

$$x = e^{1-\frac{1}{k}} \qquad y = \frac{1+k\left(1-\frac{1}{k}\right)}{e^{1-\frac{1}{k}}} = \frac{1+k-1}{e^{1-\frac{1}{k}}} = k\cdot e^{\frac{1}{k}-1}$$

$$f_k''\left(e^{1-\frac{1}{k}}\right) = \frac{2-3k+2k\cdot\left(1-\frac{1}{k}\right)}{e^{3-\frac{3}{k}}} = \frac{2-3k+2k-2}{e^{3-\frac{3}{k}}} = \frac{-k}{e^{3-\frac{3}{k}}} = \frac{<0}{>0} > 0$$

$H_k\left(e^{1-\frac{1}{k}} \middle| k\cdot e^{\frac{1}{k}-1}\right)$ ist Maximum

$H_1(1\,|\,1)$

$f_k(1) = \frac{1+k\cdot 0}{1} = 1 \quad \Rightarrow \quad$ für alle k ist $f_k(1) = 1$

c) $k = \frac{1}{2}$

NSt: $(e^{-2} \approx 0{,}1\,|\,0)$

Max: $\left(e^{-1} \approx 0{,}4 \middle| \frac{1}{2}e \approx 1{,}4\right)$

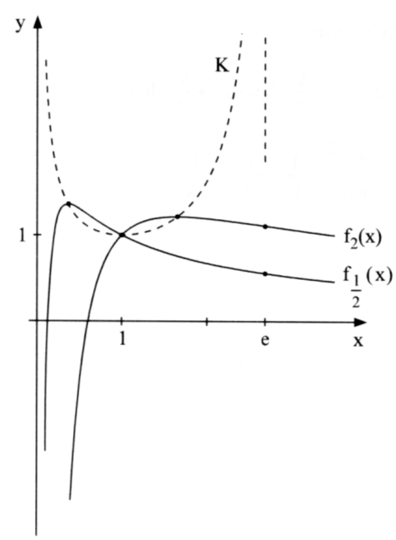
$f_{\frac{1}{2}}(e) = \frac{1{,}5}{e} \approx 0{,}55$

$k = 2$

NSt: $\left(e^{-\frac{1}{2}} \approx 0{,}6 \middle| 0\right)$

Max: $\left(e^{\frac{1}{2}} \approx 1{,}6 \middle| 2e^{-\frac{1}{2}} \approx 1{,}2\right)$

$f_2(e) = \frac{3}{e} \approx 1{,}1$



2. a) $$\left.\begin{array}{l} \lim\limits_{k \to +\infty} e^{1-\frac{1}{k}} = e^1 \\ \lim\limits_{k \overset{<}{\to} 0} e^{1-\frac{1}{k}} = "e^{-\infty}" = 0^+ \end{array}\right\} \quad x_H \in \,]0;e[$$

b) $$y_H = \frac{1}{e^{1-\frac{1}{k}} \cdot \left(1-\left(1-\frac{1}{k}\right)\right)} = \frac{1}{\frac{1}{k} \cdot e^{1-\frac{1}{k}}} = k \cdot e^{\frac{1}{k}-1} = y_H \quad \text{aus 1b}$$

$$\lim_{x_H \overset{>}{\to} 0} \frac{1}{x_H \cdot (1-\ln x_H)} = \lim_{x_H \overset{>}{\to} 0} \frac{1}{x_H - x_H \ln x_H} = "\frac{1}{0^+ - (-0^+)}" = "\frac{1}{0^+}" = +\infty$$

(s. Hinweis)

$$\lim_{x_H \overset{<}{\to} e} \frac{1}{x_H \cdot (1-\ln x_H)} = "\frac{1}{e \cdot (1-1^-)}" = "\frac{1}{0^+}" = +\infty$$

c) siehe 1c

3. $$A = \int_{e^{-\frac{1}{k}}}^{1} \frac{1+k \cdot \ln x}{x}\,dx = 1$$

$$\int \frac{1+k \cdot \ln x}{x}\,dx = \int \left(\frac{1}{x} + k \cdot \frac{\ln x}{x}\right) dx = \ln x + k \cdot \int \frac{\ln x}{x}\,dx =$$

$$= \ln x + \frac{k}{2} \cdot \int \left(2 \cdot \ln x \cdot \frac{1}{x}\right) dx = \ln x + \frac{k}{2} \cdot (\ln x)^2$$

oder mit der Substitution $x = e^z$ und somit $dx = e^z\,dz$

$$\int \frac{1+k \cdot \ln x}{x}\,dx = \int \frac{1+k \cdot z}{e^z} \cdot e^z\,dx = \int (1+kz)\,dz = z + \frac{k}{2} z^2 = \ln x + \frac{k}{2} \cdot (\ln x)^2$$

$$A = \left[\ln x + \frac{k}{2} (\ln x)^2\right]_{e^{-\frac{1}{k}}}^{1} = 0 - \left(-\frac{1}{k}\right) - \frac{k}{2}\left(-\frac{1}{k}\right)^2 = \frac{1}{k} - \frac{1}{2k} = \frac{1}{2k}$$

$$\frac{1}{2k} = 1 \quad \Rightarrow \quad k = \frac{1}{2}$$

Das können wir noch nicht